Part III

Now that we know a bit more about colimits, let's return to adjunctions.

Def An <u>adjunction between ∞-categories</u> is a pair of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

and a unit

$$\eta \in \underline{\mathrm{Fun}}(\mathcal{C}, \mathcal{C}),$$

∞-category of functors!

with $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$,

such that for every $d \in \mathcal{D}$, $c \in \mathcal{C}$, the map

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{D}}(F(c), d) \arrow[rr, "G_{F(c),d}"] \arrow[dr, "\text{composite}"'] & & \mathrm{Map}_{\mathcal{C}}(G(F(c)), G(d)) \arrow[dl, "\mathrm{Map}_{\mathcal{D}}(\eta, G(d))"] \\ & \mathrm{Map}_{\mathcal{D}}(c, G(d)) & \end{tikzcd}$$

is an equivalence of spaces

There are other equivalent definitions.

<u>Fact</u>: Quillen adjunctions go to ∞-cat. adjunctions //

<u>Lemma</u> Let $F \dashv G$ be an adjunction of ∞-cats.
Then F is an equivalence of ∞-categories
if & only if

1) F reflects equivalences $\left(x \xrightarrow[\sim]{\phi} y \Leftrightarrow Fx \xrightarrow[\sim]{F\phi} Fy \right)$

and

2) the unit η is an equivalence

~~<u>Pf</u> F is a categorical equivalence means~~

~~(a) $Ho(F)$ is a 1-categorical equivalence between $Ho(\mathcal{C})$ & $Ho(\mathcal{D})$~~

~~(b) $\forall x, y \in \mathcal{C}$, $F_{x,y} : Map_{\mathcal{C}}(x,y) \xrightarrow{\sim} Map_{\mathcal{D}}(Fx, Fy)$~~

~~The conditions (1) & (2) clearly imply (a).~~

We now want to know when we <u>can</u> get an adjoint.

<u>Prop 5.2.3.5</u> (HTT)

A left adjoint preserves colimits. No different!
A right adjoint preserves limits

To ~~state~~ state an adjoint functor theorem, we need to know a smallness condition.

The key word is <u>presentable</u> & loosely means a category is generated by a "small subcategory".

Fix a regular cardinal κ. e.g., at least $\aleph_0$

"no set of cardinality κ is the union of fewer than κ sets of cardinality less than κ"

<u>Def</u> Let <u>$\mathrm{Ind}_\kappa(\mathcal{D})$</u> denote the full subcategory of $\mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$ that is closed under κ-filtered colimits

Def An ∞-category $\mathcal{C}$ is <u>κ-accessible</u> if there is a small ∞-category $\mathcal{C}_0$ and an equivalence

$$\mathrm{Ind}_\kappa(\mathcal{C}_0) \xrightarrow{\simeq} \mathcal{C}.$$

$\begin{cases} |\pi_0 \mathcal{C}_0| < \kappa \\ |\pi_n \mathrm{Map}_{\mathcal{C}_0}(x,y)| < \kappa \quad \forall x, y \end{cases}$

An ∞-category is <u>presentable</u> if it is accessible & contains ~~all~~ small colimits.

(I have suppressed all the κ's here: κ-presentable, κ-accessible, κ-small)

<u>Theorem</u> (Cor 5.5.2.9, HTT)

Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between <u>presentable</u> ∞-categories.

only need $\mathcal{D}$ locally small!

① F has a right adjoint if & only if it preserves all small colimits

② F has a left adjoint if & only if it preserves all small limits

Now that we've gotten the important theorem on the board, let me unravel a little what presentable means by describing 1-categorical analog.

<u>Ex</u> ① Set : every set X is the colimit over the poset of its finite subsets

$$X \cong \operatorname*{colim}_{\substack{Y \subseteq X \\ |Y| < \infty}} Y \qquad \mathrm{Set} \cong \mathrm{Ind}(\mathrm{FinSet})$$

② Vect : every vector space V is the colimit over its finite dimensional subspaces

$$\mathrm{Vect} \cong \mathrm{Ind}(\mathrm{FinVect})$$

③ FinSet is <u>not</u> presentable (no <u>countable</u> coproducts)

Field is <u>not</u> ($\nexists\ \mathbb{F}_2 \amalg \mathbb{F}_3$)

Top is <u>not</u> (not sure why)

Being presentable means one can make arguments on generators & then extend by colimits

Hopefully the examples make the following plausible

together called "combinatorial"

Prop (1.3.4.22, HA)

If a model category is presentable (as a 1-cat) and cofibrantly-generated, then the underlying ∞-category is presentable.

Cor Mod_k, Lie_k, CAlg_k are all presentable

Cor Given the fact that C^*_{Lie} preserves colimits, there exists right adjoint

$$\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \longrightarrow \mathrm{Lie}_k$$